

1. (24 points) The following problems are not related. If a limit does not exist, you must say so. If you use a theorem, clearly state its name and show that its hypotheses are satisfied.

(Reminder: You may not use L'Hôpital's Rule or "Dominance of Powers" in any solutions on this exam.)

(a) $\lim_{x \rightarrow 0} \frac{\sec x}{4x \cot 2x}$

(b) $\lim_{x \rightarrow 1} \frac{\sin^2 x}{x}$

(c) $\lim_{x \rightarrow 1} \frac{x-1}{2 + \sqrt[5]{x^2}}$

Solution:

(a)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sec x}{4x \cot 2x} &= \lim_{x \rightarrow 0} \frac{1 = \cos x}{4x \frac{\cos 2x}{\sin 2x}} \\ &= \lim_{x \rightarrow 0} \frac{\sin 2x}{4x \cos x \cos 2x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin x}{4x \cos 2x} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos 2x} \\ &= \frac{1}{2} \end{aligned}$$

(b) Note that

$$0 \leq \sin^2 x \leq 1 \Rightarrow 0 \leq \frac{\sin^2 x}{x} \leq \frac{1}{x}$$

and

$$0 = \lim_{x \rightarrow 1} 0 = \lim_{x \rightarrow 1} \frac{1}{x}$$

By the Squeeze Theorem, we conclude that

$$\lim_{x \rightarrow 1} \frac{\sin^2 x}{x} = 0$$

(c)

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x-1}{2 + \sqrt[5]{x^2}} &= \lim_{x \rightarrow 1} \frac{(x-1)(2 + \sqrt[5]{x^2})}{(2 + \sqrt[5]{x^2})(2 + \sqrt[5]{x^2})} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(2 + \sqrt[5]{x^2})}{x^2 - 1} \\ &= \lim_{x \rightarrow 1} \frac{2 + \sqrt[5]{x^2}}{x+1} \\ &= \frac{2+2}{2} \\ &= 2 \end{aligned}$$

2. (21 points) The following problems are unrelated.

- (a) Given that $\csc \theta = \frac{\sqrt{5}}{2}$ and $2 < \theta < \pi$, find the values of $\tan \theta$ and $\cos(2\theta)$.
- (b) Find all values of x in the interval $[0; \pi]$ that satisfy $\tan x \sec x = 4 \sin x$.
- (c) A squirrel is up a tree, and it sees a peanut on the ground some distance away. If the straight-line distance between the peanut and the squirrel is 50 ft, and the angle between the straight-line and the tree is $\frac{\pi}{6}$ radians, how far down the tree and across the ground must the squirrel travel to reach the peanut? *Give your answer with appropriate units.*

Solution:

- (a) Since $\csc \theta = \frac{\sqrt{5}}{2}$, we know that $\sin \theta = \frac{2}{\sqrt{5}}$. Thus, the angle θ is opposite a side of length 2 in a right triangle with hypotenuse $\sqrt{5}$. The adjacent side to θ has length $\sqrt{(\sqrt{5})^2 - 2^2} = 1$. Hence, $\tan \theta = \frac{2}{1} = 2$. Using a double-angle identity for cosine, we know that

$$\cos 2\theta = 1 - 2 \sin^2 \theta = 1 - 2 \left(\frac{2}{\sqrt{5}}\right)^2 = 1 - \frac{8}{5} = -\frac{3}{5}$$

- (b) Note that

$$\begin{aligned} \tan x \sec x = 4 \sin x &\Rightarrow \frac{\sin x}{\cos^2 x} = 4 \sin x \\ &\Rightarrow \sin x - 4 \sin x \cos^2 x = 0 \\ &\Rightarrow (\sin x) (1 - 4 \cos^2 x) = 0 \end{aligned}$$

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(a) Find a formula for $f(x)$.

(b) Sketch a graph of $y = jf(x)j + 1$

(b) Note that we can cancel the $(x - 4)$ factor in the numerator and denominator of $g(x)$, so

$$g(x) = \frac{2(x - 2)}{x - 3}$$

for all x except $x = 4$. Then

$$\lim_{x \rightarrow 4} g(x) = \lim_{x \rightarrow 4} \frac{2(x - 2)}{x - 3} = \frac{2(4 - 2)}{4 - 3} = 4;$$

which shows that $x = 4$ is a removable discontinuity for $g(x)$.

Also, $x = 3$ is an infinite discontinuity (or a vertical asymptote) for $g(x)$ because

$$\lim_{x \rightarrow 3^+} g(x) = \lim_{x \rightarrow 3^+} \frac{2(x - 2)}{x - 3} = \frac{2}{\lim_{x \rightarrow 3^+} (x - 3)} = 1;$$

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(c)

$$\begin{aligned} \lim_{x \rightarrow 7} \frac{2x^2 - 12x + 16}{x^2 - 7x + 12} &= \lim_{x \rightarrow 7} \frac{x^2(2 - 12/x + 16/x^2)}{x^2(1 - 7/x + 12/x^2)} \\ &= \lim_{x \rightarrow 7} \frac{2 - 12/x + 16/x^2}{1 - 7/x + 12/x^2} \\ &= \frac{2 - 0 + 0}{1 - 0 + 0} \\ &= 2; \end{aligned}$$

By a similar argument, $\lim_{x \rightarrow 7} g(x) = 2$.

5. (10 points) Consider the function

$$f(x) = \begin{cases} b \cos(x); & x \leq 1 \\ \sqrt{2x - 2}; & x > 1 \end{cases}$$

Find the value of b such that $\lim_{x \rightarrow 1} f(x)$ exists. Justify your answer by calculating appropriate limits.

Solution: Note that

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} \sqrt{2x - 2} \\ &= \sqrt{2(1) - 2} \\ &= 0; \end{aligned}$$

So we need

$$3 = \lim_{x \rightarrow 1} b \cos(x) = b \cos(1) = b$$

for $\lim_{x \rightarrow 1} f(x)$ to exist. Hence, choosing $b = 3$ guarantees that the two-sided limit of $f(x)$ at $x = 1$ exists, in which case it equals 3.

6. (10 points) Show that the equation $x^2 - 2 = \sin x \cos x$ has at least one real solution. Indicate the interval where a solution can be found.

Solution: Let $f(x) = x^2 - 2 - \sin x \cos x$. Then the given equation has a solution where $f(x) = 0$. Note that $f(x)$ is continuous because $\sin x \cos x$ is the product of continuous functions, which is continuous, and $x^2 - 2$ is

continuous because it's a polynomial. Then $f(x)$ is given by the difference of two continuous functions, and hence is continuous itself.

Also, $f(0) = -2 < 0$, and $f(2) = 2 > 0$. Since f is continuous everywhere, in particular on $[0; 2]$, the Intermediate Value Theorem guarantees that $f(x) = 0$ has a solution in the interval $(0; 2)$, and the given equation does as well.