Name: ____________________

APPM 1350
Summer 2023

Exam 2

June 30

Instructions:

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1. (18 pts) Given the curve

$$y \sin^2(x) = \cos(x) + y^3$$

   (a) Find the derivative $\frac{dy}{dx}$ in terms of x and y.

   (b) Find the slope of the tangent line to the curve at $(\frac{}{2}; 1)$.

Solution:

(a) We want to take the derivative of the entire function with respect to the variable x.

$$\begin{aligned}
\frac{d}{dx}(y \sin^2(x)) &= \frac{d}{dx}(\cos(x) + y^3) \\
(\frac{d}{dx}y) \sin^2(x) + y(\frac{d}{dx} \sin^2(x)) &= \frac{d}{dx} \cos(x) + \frac{d}{dx}y^3 \\
\frac{dy}{dx} \sin^2(x) + y(2(\sin(x))(\cos(x)) &= -\sin(x) + 3y^2\frac{dy}{dx}
\end{aligned}$$

We then want to get all of the terms that have $\frac{dy}{dx}$ on one side of the equation, and everything else on the other side. Then we want to factor out and solve for $\frac{dy}{dx}$.

$$\begin{aligned}
\sin^2(x)\frac{dy}{dx} + 2y \sin(x) \cos(x) &= -\sin(x) + 3y^2\frac{dy}{dx} \\
\sin^2(x)\frac{dy}{dx} - 3y^2\frac{dy}{dx} &= -\sin(x) - 2y \sin(x) \cos(x) \\
\frac{dy}{dx}(\sin^2(x) - 3y^2) &= -\sin(x) - 2y \sin(x) \cos(x) \\
\frac{dy}{dx} &= \frac{-\sin(x) - 2y \sin(x) \cos(x)}{\sin^2(x) - 3y^2}
\end{aligned}$$

(b)

Thus, $L(x) = 4 - \frac{1}{8}(x - 9)$.

$$\sqrt{15.6} = g(9.4) \approx L(9.4) = 4 - \frac{1}{8}(9.4 - 9)$$
$$= 4 - \frac{1}{8}(0.4)$$
$$= 4 - \frac{1}{8}(\frac{4}{10}$$

which we can now maximize. We start by finding our critical point.

$$\begin{aligned} f'(y) &= -4y + 1 = 0 \\ -4y &= -1 \\ y &= \frac{1}{4} \end{aligned}$$

We now must verify if there is indeed a maximum at $y = \frac{1}{4}$. I will do this by using the second derivative test.
$f''(y) = -4 < 0$ for all y, therefore $y = \frac{1}{4}$ is a maximum.
Now we will use y and our given equation to find x.

$$\begin{aligned} x + 2y &= 4 \\ x + 2(\frac{1}{4}) &= 4 \\ x + \frac{1}{2} &= 4 \\ x &= 4 - \frac{1}{2} \\ x &= \frac{8}{2} - \frac{1}{2} \\ x &= \frac{7}{2} \end{aligned}$$

Therefore, the two numbers that maximize $(x + 1)(y + 2)$ are $\boxed{x = \frac{7}{2} \text{ and } y = \frac{1}{4}}$

4. (36 pts) Consider the function $y = x(x - 4)^3$,
its simplified derivative $y' = 4(x - 4)^2(x - 1)$,
and its simplified second derivative $y'' = 12(x - 4)(x - 2)$.
Justify your answers for each of the following problems.

   (a) Find the $x-$ and $y-$intercepts.

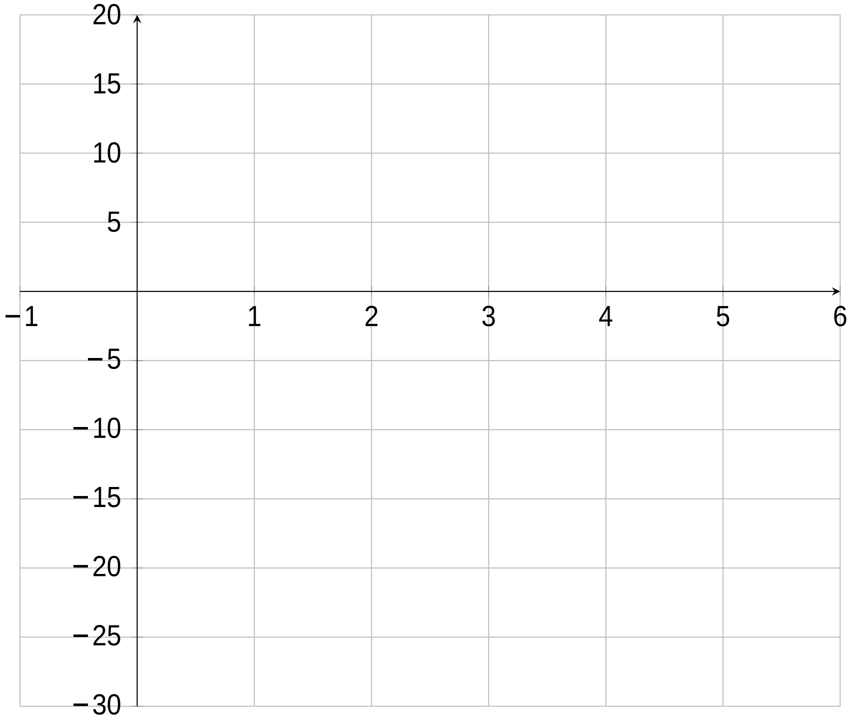
   (b) On what intervals is the function increasing? On what intervals is it decreasing?

   (c) Find the $(x; y)$ coordinates of any local maximum and local minimum values, if they exist. If none exist, state this.

   (d) On what intervals is the function concave up? On what intervals is it concave down?

   (e) Find the $(x; y)$ coordinates of any inflection points, if they exist. If none exist, state this.

   (f) Use the empty plot located on the next page to sketch the graph of this function. Carefully label all key features such as any intercepts, maximum(s), minimum(s), and inflection point(s). (Hint: there are no asymptotes for this function).



Solution:

(a) To find the $x-$intercept, we want to set $y = 0$.

$$0 = x(x-4)^3 \implies x = 0, 4$$

Therefore the $x-$ intercepts are $(0, 0), (4, 0)$. To find the $y-$intercept, we want to set $x = 0$.

$$y = 0(0-4)^3 = 0$$

Therefore the $y-$ intercept is $(0, 0)$

(b) We first need to find our critical points of the first derivative which are where it is equal to zero or where it does not exist.

$$y' = 4(x-4)^2(x-1) = 0 \implies x = 4, 1$$

We then put our critical numbers on a number line and check either side of each critical number.

- For $x < 1, \ y' < 0$
- For $1 < x < 4, \ y' > 0$
- For $x > 4, \ y' > 0$

Therefore, our function is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$

(c) We can see that y' switches from negative to positive at $x = 1$. By the first derivative test, our local minimum is where $x = 1$ and there are no local maximums.

$$y(1) = -27$$

and therefore our minimum value is at $(1, -27)$.

(d) We first need to find our critical points of the second derivative which are where it is equal to zero or where it does not exist.