

1. (40 pts) Let  $g(x; y) = x^3 - 3xy + y^3$ .

- (a) Find and classify the critical points of  $g(x; y)$ .
- (b) Find the maximum rate of change of  $g(x; y)$  at the point  $(2; 1)$  and the direction in which it occurs.
- (c) The origin and the point  $(2; 1; 3)$  lie on the surface  $z = g(x; y)$ . Find an equation for the plane that passes through the points and contains the line with symmetric equations  $x = \frac{y}{3} = z$ .
- (d) Starting at the origin, a fly takes off from the surface  $z = g(x; y)$  and travels along the path  $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 7t^2\mathbf{k}$ ,  $t \geq 0$ . At what value(s) of  $t$  will the fly meet the surface again?

**Solution:**

(a)

$$\begin{aligned} g(x; y) &= x^3 - 3xy + y^3 \\ g_x &= 3x^2 - 3y \\ g_y &= -3x + 3y^2 \end{aligned}$$

The critical points occur where  $g_x = 0$  and  $g_y = 0$ .

$$\begin{aligned} g_x = 3x^2 - 3y = 0 &\Rightarrow y = x^2 \\ g_y = -3x + 3y^2 = 0 &\Rightarrow -3x + 3x^4 = 0 \Rightarrow x = 0; 1 \end{aligned}$$

There are two critical points at  $(0; 0)$  and  $(1; 1)$ . Apply the Second Derivative Test.

$$g_{xx} = 6x \quad g_{yy} = 6y \quad g_{xy} = -3$$

$$\begin{aligned} D(x; y) &= g_{xx}g_{yy} - (g_{xy})^2 \\ D(0; 0) &= 0 \cdot 0 - (-3)^2 = -9 < 0 \\ D(1; 1) &= 6 \cdot 6 - (-3)^2 = 27 > 0 \quad \text{and} \quad g_{xx}(1; 1) = 6 > 0 \end{aligned}$$

Therefore there is a saddle point at  $g(0; 0) = 0$  and a local minimum at  $g(1; 1) = -1$ .

(b)

$$\nabla g(x; y) = \langle 3x^2 - 3y; -3x + 3y^2 \rangle$$

The gradient vector  $\nabla g(2; 1) = \langle 9; 3 \rangle$  is the direction of maximum rate of change, and the maximum rate is

$$|\nabla g(2; 1)| = \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}$$

- (c) Let  $\mathbf{v}_1 = \langle 2; 1; 3 \rangle$  be the vector connecting the two points and let  $\mathbf{v}_2 = \langle 1; 3; 1 \rangle$  be the direction vector of the line. Then a normal vector to the plane is

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix} = 8\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

and an equation of the plane is  $8x + y + 5z = 0$ .

(d) Substituting  $x = t$ ,  $y = t$ , and  $z = 7t^2$  into  $z = g(x, y)$

$$7t^2 = t^3 + 3t^2 + t^3 \Rightarrow 10t^2 = 2t^3 \Rightarrow t = 0; 5:$$

The fly begins on the surface at  $t = 0$  and meets the surface again at  $t = 5$ .

2. (15 pts) Consider the integral

$$\int_0^3 \int_{1-x}^{1+x} \frac{x-y}{x+y} dy dx$$

Use the transformation  $u = x - y$ ,  $v = x + y$  to set up an equivalent integral over a region in the  $uv$  plane. Sketch both the  $xy$  and  $uv$  regions. Do not evaluate the integral.

**Solution:**

Letting  $u = x - y$  and  $v = x + y$  gives  $x = \frac{u+v}{2}$

An equivalent integral over the  $uv$ -plane is

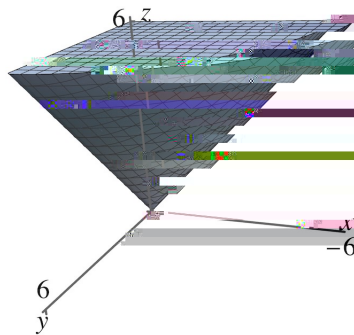
$$\int_{-1}^1 \int_{1-u}^{6-u} \frac{1}{2} \frac{u}{v} dv du \quad \text{or} \quad \int_{-1}^1 \int_{-1}^{6-v} \frac{1}{2} \frac{u}{v} du dv$$

3. (25 pts) The volume of a solid is given in cylindrical coordinates by  $\int_{-2}^2 \int_0^r \int_0^6 r dz dr d\theta$ .

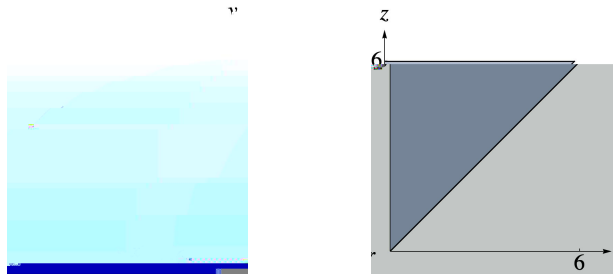
- (a) Sketch and shade the 2D cross-sections of the solid in the  $rz$ -plane (for a constant  $\theta$ ) and in the  $xy$ -plane. Label all intercepts.
- (b) Set up (but do not evaluate) an equivalent integral in rectangular coordinates in the order  $dz dy dx$ .
- (c) Set up (but do not evaluate) an equivalent integral in spherical coordinates in the order  $d\phi d\theta dr$ .

**Solution:**

The solid is a quarter cone above the second quadrant of the  $xy$ -plane, bounded below by  $z = r = \sqrt{x^2 + y^2}$  and above by the plane  $z = 6$ .



(a)



(b) In rectangular coordinates, an equation for the cone is  $z = \sqrt{x^2 + y^2}$ . A semicircle of radius 6 centered at the origin has the equation  $y = \sqrt{36 - x^2}$ .

4. (25 pts)

(a) Use Gaussian elimination to solve the linear system.

$$\begin{aligned}2x + 4y &= 10 \\x + 4y + z &= 6 \\x + y &= 4\end{aligned}$$

(b) Reduce this homogeneous system to RREF and use the result to find the complete solution set.

$$\begin{aligned}2x + 4y &= 0 \\x + 4y + z &= 0\end{aligned}$$

**Solution:**

(a) First row reduce the augmented matrix

$$\begin{array}{cccc|ccc} & 2 & & & & & & \\ & 1 & 0 & 0 & \frac{1}{3} & 1 & 1 & \\ ! & 40 & 1 & 0 & \frac{1}{3} & 0 & 1 & \\ & 0 & 0 & 1 & & & & \end{array}$$

