Write your name and your professor's name or your section number in the top right corner of your paper. You are allowed to use textbooks and notes, but you may not ask anyone for help except the proctors. To receive full credit on a problem you must show **su cient justi cation for your conclusion** unless explicitly stated otherwise.

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1. (30 points: 3 each) If the statement is **always true** mark \TRUE"; if it is possible for the statement to be false then mark \FALSE." **No justi cation is necessary.**

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2. (15 points) Suppose that $A\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix}$ and $A\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\3\end{pmatrix}$. Solve $Ax = \begin{pmatrix}1\\0\\6\end{pmatrix}$.

Solution: First solve

$$c_1\begin{pmatrix}1\\2\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\3\end{pmatrix} = \begin{pmatrix}1\\0\\6\end{pmatrix}.$$

The solution can be obtained by looking at the first and last equations, and is $c_1 = 1$ $c_2 = 2$. Next find the solution x as

$$c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} = x.$$

3. (20 points) Prove that the following expression defines an inner product:

$$\langle u, v\rangle = 2u_1(v_1 \quad 2v_2) + u_2(\quad 4v_1 + 9v_2 + v_3) + u_3(v_2 + 4v_3)$$

Solution: We will show that $\langle u, v\rangle = u^T K v$ where K is a symmetric positive definite (SPD) matrix. This proves that the formula defines an inner product.

First, note that

$$K = \begin{bmatrix}2 & 4 & 0\\ 4 & 9 & 1\\ 0 & 1 & 4\end{bmatrix}.$$

Clearly this is symmetric. Row reducing never requires a row permutation, and leads to the REF

$$\begin{bmatrix}2 & 4 & 0\\ 0 & 1 & 1\\ 0 & 0 & 3\end{bmatrix}.$$

The diagonal entries are all positive, so the matrix is SPD.

4. (20 points) Find the QR factorization of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ using the Gram-Schmidt version of QR.

Solution:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \frac{4}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{0}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$$

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